JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 87, 489–508 (1982)

A Boundary Value Technique for Singular Perturbation Problems

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A new way to solve singular perturbation problems is introduced. It is designed for the practicing engineer or applied mathematician who needs a practical tool for these problems (easy to use, modest problem preparation, and ready computer implementation). As with other methods, the original problem is partitioned into inner and outer solution differential equation systems. However, asymptotic solution techniques are not employed. The method is distinguished by the following facts: The inner solution problem is solved as a two-point boundary problem, where the terminal boundary conditions are supplied by the solution of the outer solution problem. In turn, the outer solution differential equations are solved as an initial value system, where the initial conditions are supplied by the inner solution at its terminal point. The method is iterative on the terminal point of the inner solution problem. Three numerical examples are included.

1. INTRODUCTION

A wide variety of techniques have been applied to the solution of singular perturbation problems [1–7, 11]. In one form or another these techniques consist of (1) dividing the problem into an inner solution (or boundary layer) problem and an outer solution problem, (2) expressing the inner and outer solutions as asymptotic expansions, (3) equating various terms in the inner and outer solution expressions to determine the constants in these expressions, and (4) combining the inner and outer solutions in some fashion to obtain a uniformly valid solution. Typically the inner solution equations are obtained from the original differential equation by rescaling the independent variable. In Lighthill's method [3–5], however, both the dependent and independent variables are transformed. Commonly the differential equations for the outer solution are obtained by setting ε, the perturbation parameter, to zero. When this is done for the familiar problems where ε multiplies the highest derivative, the order of the system is reduced. Consequently, the analyst must determine which of the boundary conditions apply to the differential equation of the outer solution.

For the engineer or applied mathematician who is not a specialist in

0022-247X/82/060489-20$02.00/0

perturbation, the vast literature and multiple techniques for solving singular perturbation problems represent quite a challenge. Such matters as finding the appropriate asymptotic expansions are not routine exercises but require skill, insight, and experimentation. Even the matching of the coefficients of the inner and outer solution expansions can be a demanding process.

The purpose of this paper is to present a technique for solving singular perturbation problems that does not depend on asymptotic expansions and the matching of coefficients. It is designed primarily for the practicing engineer and applied mathematician who need a practical tool for solving singular perturbation problems. (See [8–10] for techniques, designed for similar purposes, applicable to regular perturbation problems.) The method requires a minimum of problem preparation and is readily implemented on a computer. Numerical experience with three examples is reported.

2. Boundary Value Technique

For convenience we call our method the "boundary value technique" and refer to the other methods which match coefficients of asymptotic expansions as "coefficient matching" methods.

In the boundary value technique we partition the original singular perturbation problem into two problems, an inner solution differential equation problem and an outer solution differential equation problem. The solution of the outer solution differential equation provides the terminal conditions for the inner solution differential equation problem. And in turn, the solution of the inner solution differential equation provides the initial conditions for the solution of the original singular perturbation problem starting at the terminal point of the inner solution. The problem is solved iteratively for various values of the terminal point and the terminal boundary conditions of the inner solution problem until the profiles stabilize and the boundary conditions of the original problem are satisfied.

To fix the ideas consider the following singular perturbation problem

$$\varepsilon y''(x) + y'(x) + y(x) = 0, \qquad 0 \leqslant x \leqslant 1, \tag{2.1}$$

$$y(0) = \alpha, \qquad y(1) = \beta. \tag{2.2}$$

The boundary value method consists of the following steps:

1. Convert the original singular perturbation equation to the differential equation of the outer solution and determine which boundary conditions applies. Setting $\varepsilon = 0$, (2.1) reduces to the outer solution differential equation

$$y'(x) + y(x) = 0 \tag{2.3}$$

with

$$y(1) = \beta \tag{2.4}$$

as the appropriate boundary condition.

2. From the original singular perturbation equation, determine the scaling equation to create the inner solution differential equation. Here set

$$t = x/\varepsilon. \tag{2.5}$$

3. Using (2.5), rescale (2.1) with

$$y(x) = Y(t), \tag{2.6a}$$

$$y'(x) = Y'(t)/\varepsilon, \tag{2.6b}$$

$$y''(x) = Y''(t)/\varepsilon^2 \tag{2.6c}$$

to obtain the differential equation of the inner solution

$$Y''(t) + Y'(t) + \varepsilon Y(t) = 0. \tag{2.7}$$

4. Determine the boundary conditions for the inner solution differential equation. By (2.5) and (2.6a)

$$Y(0) = y(0) = \alpha. \tag{2.8}$$

Choose t_f to be the terminal point or "width" or "thickness" of the inner solution. Solve the outer solution differential equations (2.3)–(2.4), over the interval $t_f\varepsilon \leqslant x \leqslant 1$, to obtain $y(t_f\varepsilon)$:

$$Y(t_f) = y(t_f\varepsilon) = \hat{\beta}. \tag{2.9}$$

5. Solve the inner solution differential equation (2.7) with the two-point boundary conditions (2.8)–(2.9) over the interval $0 \leqslant t \leqslant t_f$ to obtain the inner solution, $Y(t)$. Express $Y(t)$ also as $y(t\varepsilon) = y(x)$, $0 \leqslant x \leqslant t_f\varepsilon$.

6. Integrate the original singular perturbation problem (2.1) as an initial value problem with the initial conditions

$$y(t_f\varepsilon) = Y(t_f) = \hat{\beta}, \tag{2.10}$$

$$y'(t_f\varepsilon) = Y'(t_f)/\varepsilon \tag{2.11}$$

over the interval $t_f\varepsilon \leqslant x \leqslant 1$.

7. Adjoin the solutions of items 5 and 6 at $t_f\varepsilon = x$ to obtain the trial solution of (2.1)–(2.2) over $0 \leqslant x \leqslant 1$.

8. Repeat the process from steps 4 through 7 for $t_f^{(k)}$, $k = 1, 2, \ldots,$

(where $t_f^{(k)}$ is the value of the t_f for the kth iteration), until the solution profiles do not differ materially from iteration to iteration and boundary conditions (2.2) are satisfied.

As an alternative to item 6, we may use the solution of the outer solution differential equation (2.3)–(2.4) over the interval $t_f\varepsilon \leqslant x \leqslant 1$.

To carry out item 8 we may use either relative or absolute error criteria. For relative error we require that

$$\left| \frac{Y(t)^{(k+1)} - Y(t)^{(k)}}{Y(t)^{(k)}} \right| \leqslant \delta_1, \qquad 0 \leqslant t \leqslant t_f, \tag{2.12}$$

where

$$Y(t)^{(k)} = k\text{th iterate of the inner solution},$$

$$\delta_1 = \text{prescribed tolerance bound}.$$

For absolute error criterion, we employ

$$|Y(t)^{(k+1)} - Y(t)^{(k)}| \leqslant \delta_2, \qquad 0 \leqslant t \leqslant t_f, \tag{2.13}$$

where

$$\delta_2 = \text{prescribed tolerance bound}.$$

3. Comments

The gist of the boundary value method is to find the terminal boundary conditions for the inner solution which match the initial conditions for the outer solution. The search for the "proper" t_f is the vehicle for doing this. Theoretically the original singular perturbation problem (2.1)–(2.2) can be solved by the inner solution differential equation (2.7) over the interval $0 \leqslant t \leqslant 1/\varepsilon$ with the boundary conditions $Y(0) = \alpha$, $Y(1/\varepsilon) = \beta$. Practically, the interval $[0, 1/\varepsilon]$ becomes unreasonably large as $\varepsilon \to 0$ so we limit the range to $[0, t_f]$, where $t_f \leqslant 1/\varepsilon$. The t_f is not unique but can assume a wide range of values. To reduce the amount of computation we desire the smallest value of t_f that gives the required accuracy. Looking at t_f another way, we may say that t_f is that value at which the term $\varepsilon y''$ in (2.1) becomes negligibly small, so that the outer solution differential equation generates the same solution as the original differential equation would over the interval $[t_f\varepsilon, 1]$.

Because the inner solution interval is small relative to the entire interval of the original problem, we can usually improve our accuracy by making t_f

larger. The analyst must balance his need for accuracy against the costs of larger running times.

In our prescription for the boundary value method we have assumed that the inner and outer solutions can be found. While our orientation is toward numerical methods using a computer, there is no reason that other solution techniques cannot be used: analytical, approximation, or even asymptotic methods.

The boundary value technique is similar in some respects to the coefficient matching methods in that inner and outer solution differential equations are used and the two solutions fused. The methods differ in how they use the data which are available. We determine boundary conditions at t_f as the mechanism for generating the solution of the original problem from the inner and outer solution differential equations. On the other hand, the coefficient matching methods, as the name implies, determines the values of the coefficients which in turn are used in the asymptotic expressions to generate the solution.

The issue of the selection of the boundary conditions for the inner and outer solution differential equations is inherent in all inner–outer solution methods. Furthermore, the resolution of the boundary conditions is essentially the same for all inner–outer solution methods, including this method. The choice of the boundary conditions is often made by inspection, insight, experience, experimentation, or knowledge of the physics of the problem. Cole [2, pp. 29–32] and O'Malley [7, pp. 45–59] provide useful approaches. In the examples in this paper we employed Cole's method and experience.

4. Numerical Examples

To illustrate the boundary value method, we have applied it to three singular perturbation problems: a linear system with constant coefficients, a linear system with variable coefficients, and a nonlinear system. Each of these examples has been chosen because either analytical or approximate solutions are available for comparison.

Example 1 [7, pp. 6–7, 18–21].

$$\varepsilon y''(x) + y'(x) + y(x) = 0, \qquad 0 \leqslant x \leqslant 1, \tag{4.1}$$

$$y(0) = \alpha, \qquad y(1) = \beta. \tag{4.2}$$

The exact solution is given by

$$y(x) = \left(\frac{\beta - \alpha e^{\rho_2}}{e^{\rho_1} - e^{\rho_2}}\right) e^{\rho_1 x} + \left(\frac{\alpha e^{\rho_1} - \beta}{e^{\rho_1} - e^{\rho_2}}\right) e^{\rho_2 x}. \tag{4.3}$$

where

$$\rho_1 = \frac{-1 + \sqrt{1-4\varepsilon}}{2\varepsilon}, \qquad \rho_2 = \frac{-1 - \sqrt{1-4\varepsilon}}{2\varepsilon}. \tag{4.4}$$

The outer solution problem is expressed by

$$y'(x) + y(x) = 0, \tag{4.5}$$

$$y(1) = \beta, \tag{4.6}$$

whose solution is

$$y(x) = \beta e^{1-x}. \tag{4.7}$$

Using the scaling

$$t = x/\varepsilon \tag{4.8}$$

we write the inner solution problem as

$$Y''(t) + Y'(t) + \varepsilon Y(t) = 0, \qquad 0 \leqslant t \leqslant t_f, \tag{4.9}$$

$$Y(0) = \alpha, \qquad Y(t_f) = \hat{\beta}, \tag{4.10}$$

where from (4.7)

$$\hat{\beta} = \beta e^{1-t_f\varepsilon}. \tag{4.11}$$

The analytical solution to (4.9)–(4.10) is given by

$$Y(t) = \left(\frac{\hat{\beta} - \alpha e^{r_2 t_f}}{e^{r_1 t_f} - e^{r_2 t_f}}\right) e^{r_1 t} + \left(\frac{-\hat{\beta} + \alpha e^{r_1 t_f}}{e^{r_1 t_f} - e^{r_2 t_f}}\right) e^{r_2 t}, \tag{4.12}$$

where

$$r_1 = \frac{-1 + \sqrt{1-4\varepsilon}}{2}, \qquad r_2 = \frac{-1 - \sqrt{1-4\varepsilon}}{2}. \tag{4.13}$$

In Tables 1A, 1B, and 1C we have tabulated for $\alpha = 1$, $\beta = 2$, and $\varepsilon = 10^{-3}$, 10^{-4}, and 10^{-6} the solutions obtained by the boundary value technique as well as the exact solution. For each value of ε, the solutions are listed at $t_f = 1$, 10, 20, and 30. Upon scanning the tables we observe for each ε that with the exception of the solution at $t_f = 1$, all the solutions compare favorably to one another and to the exact solution. We may choose any value of $t_f \geqslant 10$ as the "width" of the inner solution. In the last column of the tables we have given the relative error of the solution at $t_f = 30$ to the exact

TABLE 1A

Example 1, $\varepsilon = 10^{-3}$

$t_f \rightarrow$	1	10	20	30		Absolute Rel. Error at $t_f=30$
x	y(x)	y(x)	y(x)	y(x)	$y(x)_{exact}$	
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	0.0
$2.5(10^{-4})$	2.5509322	1.9791967	1.9791634	1.9791754	1.9803425	.00058
$5.0(10^{-4})$	3.7586584	2.7416864	2.7416271	2.7416485	2.7437244	.00076
$1.0(10^{-3})$	5.4311298	3.7975332	3.7974380	3.7974723	3.8008070	.00087
$5.0(10^{-3})$		5.3796352	5.3794859	5.3795398	5.3847643	.00097
$1.0(10^{-2})$		5.3824689	5.3823194	5.3823734	5.3876072	.00097
$2.0(10^{-2})$			5.3289125	5.3289659	5.3341478	.00097
$3.0(10^{-2})$				5.2758889	5.2810193	.00097
$4.0(10^{-2})$					5.2284199	.00097
$5.0(10^{-2})$	7.6348927	5.1714075	5.1712639	5.1713157	5.1763444	.00097
$1.0(10^{-1})$	7.2621708	4.9189486	4.9188119	4.9188612	4.9236444	.00097
$2.0(10^{-1})$	6.5704255	4.4504028	4.4502791	4.4503237	4.4546513	.00097
$3.0(10^{-1})$	5.9445711	4.0264875	4.0263756	4.0264160	4.0303313	.00097
$4.0(10^{-1})$	5.3783315	3.6429515	3.6428503	3.6428868	3.6464292	.00097
$5.0(10^{-1})$	4.8660279	3.2959485	3.2958570	3.2958900	3.2990950	.00097
$6.0(10^{-1})$	4.4025230	2.9819987	2.9819159	2.9819458	2.9848455	.00097
$7.0(10^{-1})$	3.9831684	2.6979537	2.6978788	2.6979058	2.7005293	.00097
$8.0(10^{-1})$	3.6037587	2.4409649	2.4408971	2.4409215	2.4432951	.00097
$9.0(10^{-1})$	3.2604890	2.2084550	2.2083937	2.2084158	2.2105633	.00097
1.0	2.9499169	1.9980925	1.9980370	1.9980571	2.0000000	.00097

TABLE 1B

Example 1, $\varepsilon = 10^{-4}$

$t_f \rightarrow$	1	10	20	30		Absolute Rel. Error at $t_f=30$
x	y(x)	y(x)	y(x)	y(x)	$y(x)_{exact}$	
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	0.0
$2.5(10^{-5})$	2.5523392	1.9811868	1.9811423	1.9811425	1.9812624	.000061
$5.0(10^{-5})$	3.7612881	2.7453250	2.7452458	2.7452460	2.7454593	.000077
$1.0(10^{-4})$	5.4360200	3.8038641	3.8037369	3.8037372	3.8040798	.000090
$1.0(10^{-3})$		5.4311298	5.4309287	5.4309293	5.4314709	.000099
$2.0(10^{-3})$			5.4257014	5.4257019	5.4262430	.000099
$3.0(10^{-3})$				5.4202784	5.4208189	.000099
$4.0(10^{-3})$					5.4154003	.000099
$5.0(10^{-3})$					5.4099871	.000099
$5.0(10^{-2})$	7.6282108	5.1715860	5.1713945	5.1713950	5.1719107	.000099
$1.0(10^{-1})$	7.2561423	4.9193401	4.9191580	4.9191585	4.9196491	.000099
$2.0(10^{-1})$	6.5655634	4.4511585	4.4509937	4.4509942	4.4514380	.000099
$3.0(10^{-1})$	5.9407080	4.0275345	4.0273854	4.0273858	4.0277874	.000099
$4.0(10^{-1})$	5.3753211	3.6442275	3.6440925	3.6440929	3.6444563	.000099
$5.0(10^{-1})$	4.8637430	3.2974004	3.2972783	3.2972786	3.2976075	.000099
$6.0(10^{-1})$	4.4008527	2.9835814	2.9834709	2.9834712	2.9837688	.000099
$7.0(10^{-1})$	3.9820163	2.6996291	2.6995291	2.6995294	2.6997986	.000099
$8.0(10^{-1})$	3.6030413	2.4427010	2.4426106	2.4426108	2.4428544	.000099
$9.0(10^{-1})$	3.2601340	2.2102251	2.2101423	2.2101435	2.2103639	.000099
1.0	2.9498617	1.9998744	1.9998804	1.9998006	2.0000000	.000099

TABLE 1C

Example 1, $\varepsilon = 10^{-6}$

$t_f \rightarrow$	1	10	20	30		Absolute Rel. Error at $t_f=30$
x	y(x)	y(x)	y(x)	y(x)	$y(x)_{exact}$	
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	0.0
$2.5(10^{-7})$	2.5524941	1.9814067	1.9813622	1.9813622	1.9813634	.0000006
$5.0(10^{-7})$	3.7615775	2.7457270	2.7456477	2.7456477	2.7456499	.0000008
$1.0(10^{-6})$	5.4365582	3.8045634	3.8044360	3.8044360	3.8044395	.0000009
$2.0(10^{-6})$		4.8363022	4.8361280	4.8361280	4.8361327	.0000010
$5.0(10^{-6})$		5.4068431	5.4066430	5.4066430	5.4066484	.0000010
$1.0(10^{-5})$		5.4365093	5.4363079	5.4363079	5.4363133	.0000010
$2.0(10^{-5})$			5.4364549	5.4364549	5.4364604	.0000010
$3.0(10^{-5})$				5.4364006	5.4364060	.0000010
$4.0(10^{-5})$					5.4363516	
$5.0(10^{-5})$					5.4362973	
$5.0(10^{-2})$	7.6274789	5.1716107	5.1714191	5.1714191	5.1714242	.0000010
$1.0(10^{-1})$	7.2554820	4.9193880	4.9192057	4.9192057	4.9192106	.0000010
$2.0(10^{-1})$	6.5650031	4.4512459	4.4510810	4.4510810	4.4510854	.0000010
$3.0(10^{-1})$	5.9402851	4.0276534	4.0275042	4.0275042	4.0275082	.0000010
$4.0(10^{-1})$	5.3749917	3.6443712	3.6442361	3.6442361	3.6442398	.0000010
$5.0(10^{-1})$	4.8634931	3.2975631	3.2974409	3.2974409	3.2974442	.0000010
$6.0(10^{-1})$	4.4006701	2.9837582	2.9836476	2.9836476	2.9836506	.0000010
$7.0(10^{-1})$	3.9818906	2.6998158	2.6997157	2.6997157	2.6997184	.0000010
$8.0(10^{-1})$	3.6029632	2.4428941	2.4428036	2.4428036	2.4428060	.0000010
$9.0(10^{-1})$	3.2600956	2.2104217	2.2103399	2.2103398	2.2103421	.0000010
1.0	2.9498562	2.0000721	1.9999980	1.9999980	2.0000000	.0000010

solution. For $\varepsilon = 10^{-3}$, 10^{-4}, and 10^{-6}, the maximum absolute relative errors are 9.7 (10^{-4}), $9.9(10^{-5})$, and 10^{-6}, respectively. Interestingly the smallest error occurs near the origin, and the relative error remains constant and equal to the maximum relative error over almost the entire interval.

EXAMPLE 2 [6, pp. 148–150].

$$\varepsilon y''(x) + (2x+1)\,y'(x) + 2y(x) = 0, \qquad 0 \leqslant x \leqslant 1, \tag{4.14}$$

$$y(0) = \alpha = 1, \qquad y(1) = \beta = 2. \tag{4.15}$$

While (4.14) is an exact second order equation we have chosen to use Nayfeh's uniformly valid approximation as our "exact" solution [6, p. 149, Eq. (4.2.41)].

$$y(x) = \frac{3\beta}{2x+1} + (\alpha - 3\beta)\, e^{-(x^2+x)/\varepsilon} + O(\varepsilon). \tag{4.16}$$

The outer solution problem is given by

$$(2x+1)\,y'(x) + 2y(x) = 0, \tag{4.17}$$

$$y(1) = \beta, \tag{4.18}$$

whose solution is

$$y(x) = 3\beta/(2x+1). \tag{4.19}$$

Using the scaling $t = x/\varepsilon$, we write the inner solution problem as

$$Y''(t) + (2\varepsilon t + 1)\, Y'(t) + 2\varepsilon Y(t) = 0, \tag{4.20}$$

$$Y(0) = \alpha = 1, \qquad Y(t_f) = \hat{\beta}, \tag{4.21}$$

where from (4.19)

$$\hat{\beta} = 3\beta/(2\varepsilon t_f + 1). \tag{4.22}$$

Because (4.20) is an exact second order equation, we may write immediately a first integral

$$Y'(t) + (2\varepsilon t + 1)\, Y(t) = C_1. \tag{4.23}$$

Integrating (4.23) yields

$$Y(t) = e^{-(\varepsilon t^2 + t)} C_1 \int^{t} e^{(\varepsilon t^2 + t)}\, dt + C_2 e^{-(\varepsilon t^2 + t)}. \tag{4.24}$$

Upon approximating the integral by expanding the exponential up through first order terms and then integrating, we have

$$Y(t) = C_1 e^{-\epsilon t^2}[1 + \varepsilon(t^2 - 2t + 2)] + C_2 e^{-(\epsilon t^2 + t)}. \tag{4.25}$$

On invoking the boundary conditions (4.21), we evaluate C_1 and C_2 as

$$C_1 = \frac{\hat{\beta} - \alpha e^{-(\epsilon t_f^2 + t_f)}}{e^{-\epsilon t_f^2}[1 + \varepsilon(t_f^2 - 2t_f + 2)] - e^{-(\epsilon t_f^2 + t_f)}(1 + 2\varepsilon)}, \tag{4.26}$$

$$C_2 = \alpha - C_1(1 + 2\varepsilon). \tag{4.27}$$

In Tables 2A, 2B, and 2C we have tabulated for $\alpha = 1$, $\beta = 2$, and $t_f = 1, 10, 20$, and 30 the solutions obtained by the boundary value method for $\varepsilon = 10^{-3}$, 10^{-4}, and 10^{-6}, respectively. The last two columns of the tables list the exact solution and the absolute relative errors to the exact solution at the t_f specified. As expected, for small ε, $\varepsilon = 10^{-6}$ (see Table 2C), the boundary value method solutions for each t_f (except $t_f = 1$) are close to each other and converge to the exact solution as t_f increases. For $\varepsilon = 10^{-6}$ and $t_f = 30$, the maximum relative error is $4(10^{-7})$.

For larger ε, such as $\varepsilon = 10^{-3}$, we observe that the profiles in Table 2A as a function of t_f are not close to each other and do not converge to the exact solution in the inner solution interval. Examination of $y(x)$, C_1, and C_2 reveals that they take a minimum value at $t_f = 10$ and furthermore at $t_f = 10$ the $y(x)$ profile closely approximates the exact solution. In Table 2B for $\varepsilon = 10^{-4}$ we find that with the exception of the profile at $t_f = 1$, the profiles at the various t_f are close to each other. Viewing the profiles as a function of t_f, we observe the profile at $t_f = 10$ is the minimum. It closely approximates the exact solution and exhibits a maximum relative error of $7(10^{-5})$. If we examine C_1 in (4.26) we observe that as $\varepsilon \to 0$, $C_1 \to \hat{\beta}$, which our numerical experience confirms. Because the method depends on ε being small, it appears for this problem that $\varepsilon = 10^{-4}$ is the upper bound.

Example 3 [7, p. 117]. This is a pathological case. It is particularly difficult to solve by any method since its solution for the boundary conditions given here drops instantaneously from the initial value to zero and remains at zero for the entire interval. The equation has the interesting property that one of the constants in the analytical solution is a complex number.

$$\varepsilon y''(x) + y'(x) - y'(x)^2 = 0, \qquad 0 \leqslant x \leqslant 1, \tag{4.28}$$

$$y(0) = \alpha = 1, \qquad y(1) = \beta = 0. \tag{4.29}$$

TABLE 2A

Example 2, $\varepsilon = 10^{-3}$

$t_f \rightarrow$	1	10	20	30		Absolute Rel. Error at $t_f=10$
x	y(x)	y(x)	y(x)	y(x)	$y(x)_{exact}$	
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	0.0
$2.5(10^{-4})$	2.7466150	2.1077814	2.1764865	2.4505619	2.1032410	0.0021
$5.0(10^{-4})$	4.1064098	2.9701837	3.0923823	3.5798513	2.9621108	0.0027
$1.0(10^{-3})$	5.9880240	4.1634192	4.3596514	5.1424518	4.1504652	0.0031
$2.0(10^{-3})$		5.3197713	5.5878124	6.6570695	5.3021205	0.0033
$5.0(10^{-3})$		5.9268125	6.2328636	7.4537486	5.9077361	0.0032
$1.0(10^{-2})$		5.8823529	6.1864433	7.3995065	5.8821475	0.00003
$2.0(10^{-2})$			5.7692308	6.9004928	5.7692308	
$3.0(10^{-2})$				5.6603774	5.6603774	
$4.0(10^{-2})$					5.5555556	
$5.0(10^{-2})$	5.4545455	5.4545455	5.4545455	5.4545455	5.4545455	0.0
$1.0(10^{-1})$	5.0000000	5.0000000	5.0000000	5.0000000	5.0000000	0.0
$2.0(10^{-1})$	4.2857143	4.2857143	4.2857143	4.2857143	4.2857143	0.0
$3.0(10^{-1})$	3.7500000	3.7500000	3.7500000	3.7500000	3.7500000	0.0
$4.0(10^{-1})$	3.3333333	3.3333333	3.3333333	3.3333333	3.3333333	0.0
$5.0(10^{-1})$	3.0000000	3.0000000	3.0000000	3.0000000	3.0000000	0.0
$6.0(10^{-1})$	2.7272727	2.7272727	2.7272727	2.7272727	2.7272727	0.0
$7.0(10^{-1})$	2.5000000	2.5000000	2.5000000	2.5000000	2.5000000	0.0
$8.0(10^{-1})$	2.3076923	2.3076923	2.3076923	2.3076923	2.3076923	0.0
$9.0(10^{-1})$	2.1428571	2.1428571	2.1428571	2.1428571	2.1428571	0.0
1.0	2.0000000	2.0000000	2.0000000	2.0000000	2.0000000	0.0

TABLE 2B

Example 2, $\varepsilon = 10^{-4}$

$t_f \rightarrow$	1	10	20	30		Absolute Rel. Error at $t_f=10$
x	y(x)	y(x)	y(x)	y(x)	$y(x)_{exact}$	
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	0.0
$2.5(10^{-5})$	2.7493549	2.1058173	2.1065803	2.1102071	2.1057204	.000046
$5.0(10^{-5})$	4.1117069	2.9669948	2.9683521	2.9748033	2.9668226	.000058
$1.0(10^{-4})$	5.9988002	4.1598635	4.1620440	4.1724077	4.1595870	.000066
$2.0(10^{-4})$		5.3215730	5.3245551	5.3387294	5.3211952	.000071
$5.0(10^{-4})$		5.9608232	5.9642468	5.9805195	5.9604004	.000070
$1.0(10^{-3})$		5.9880240	5.9914670	6.0078324	5.9877992	.000037
$2.0(10^{-3})$			5.9760956	5.9924191	5.9760956	
$3.0(10^{-3})$				5.9642147	5.9642147	
$4.0(10^{-3})$					5.9523810	
$5.0(10^{-2})$	5.4545455	5.4545455	5.4545455	5.4545455	5.4545455	0.0
$1.0(10^{-1})$	5.0000000	5.0000000	5.0000000	5.0000000	5.0000000	0.0
$2.0(10^{-1})$	4.2857143	4.2857143	4.2857143	4.2857143	4.2857143	0.0
$3.0(10^{-1})$	3.7500000	3.7500000	3.7500000	3.7500000	3.7500000	0.0
$4.0(10^{-1})$	3.3333333	3.3333333	3.3333333	3.3333333	3.3333333	0.0
$5.0(10^{-1})$	3.0000000	3.0000000	3.0000000	3.0000000	3.0000000	0.0
$6.0(10^{-1})$	2.7272727	2.7272727	2.7272727	2.7272727	2.7272727	0.0
$7.0(10^{-1})$	2.5000000	2.5000000	2.5000000	2.5000000	2.5000000	0.0
$8.0(10^{-1})$	2.3076923	2.3076923	2.3076923	2.3076923	2.3076923	0.0
$9.0(10^{-1})$	2.1428571	2.1428571	2.1428571	2.1428571	2.1428571	0.0
1.0	2.0000000	2.0000000	2.0000000	2.0000000	2.0000000	0.0

TABLE 2C

Example 2, $\varepsilon = 10^{-6}$

$t_f \rightarrow$	1	10	20	30		Absolute Rel. Error at $t_f=30$
x	y(x)	y(x)	y(x)	y(x)	$y(x)_{exact}$	
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	0.0
$2.5(10^{-7})$	2.7496570	2.1060435	2.1059934	2.1059938	2.1059933	.0000002
$5.0(10^{-7})$	4.1122908	2.9674308	2.9673416	2.9673423	2.9673415	.0000003
$1.0(10^{-6})$	5.9999880	4.1607361	4.1605929	4.1605940	4.1605926	.0000003
$2.0(10^{-6})$		5.3234986	5.3233026	5.3233041	5.3233023	.0000003
$5.0(10^{-6})$		5.9664766	5.9662515	5.9662532	5.9662511	.0000002
$1.0(10^{-5})$		5.9998800	5.9996534	5.9996551	5.9996530	.0000004
$2.0(10^{-5})$			5.9997600	5.9997617	5.9997600	.0000003
$3.0(10^{-5})$				5.9996400	5.9996400	0.0
$4.0(10^{-5})$					5.9995200	
$5.0(10^{-2})$	5.4545455	5.4545455	5.4545455	5.4545455	5.4545455	0.0
$1.0(10^{-1})$	5.0000000	5.0000000	5.0000000	5.0000000	5.0000000	0.0
$2.0(10^{-1})$	4.2857143	4.2857143	4.2857143	4.2857143	4.2857143	0.0
$3.0(10^{-1})$	3.7500000	3.7500000	3.7500000	3.7500000	3.7500000	0.0
$4.0(10^{-1})$	3.3333333	3.3333333	3.3333333	3.3333333	3.3333333	0.0
$5.0(10^{-1})$	3.0000000	3.0000000	3.0000000	3.0000000	3.0000000	0.0
$6.0(10^{-1})$	2.7272727	2.7272727	2.7272727	2.7272727	2.7272727	0.0
$7.0(10^{-1})$	2.5000000	2.5000000	2.5000000	2.5000000	2.5000000	0.0
$8.0(10^{-1})$	2.3076923	2.3076923	2.3076923	2.3076923	2.3076923	0.0
$9.0(10^{-1})$	2.1428571	2.1428571	2.1428571	2.1428571	2.1428571	0.0
1.0	2.0000000	2.0000000	2.0000000	2.0000000	2.0000000	0.0

The exact solution is given by

$$y(x) = -\varepsilon \ln(1 + e^{-1/\epsilon} - e^{-x/\epsilon}). \tag{4.30}$$

The outer solution problem is given by

$$y'(x)[1 - y'(x)] = 0. \tag{4.31}$$

Two possible outer solution differential equations exist.

$$y'(x) = 0, \tag{4.32}$$

$$y'(x) = 1. \tag{4.33}$$

Equation (4.32) is the correct outer solution differential equation with $y(1) = 0$ as the terminal condition. Integrating (4.32) yields the outer solution

$$y(x) = 0, \qquad t_f \varepsilon \leqslant x \leqslant 1. \tag{4.34}$$

Using the scaling

$$t = x/\varepsilon, \tag{4.35}$$

we write the inner solution problem as

$$Y''(t) + Y'(t) - (Y'(t))^2/\varepsilon = 0, \tag{4.36}$$

$$Y(0) = 1, \qquad Y(t_f) = \hat{\beta}, \tag{4.37}$$

where

$$\hat{\beta} = y(t_f \varepsilon) = 0, \tag{4.38}$$

Equation (4.36) may be solved analytically to give

$$Y(t) = \varepsilon \ln \varepsilon(-e^{t\varepsilon + C_3^{\mathrm{R}}} - C_1)^{-1}. \tag{4.39}$$

$$C_1 = \frac{\varepsilon(e^{(\hat{\beta} - t_f\varepsilon - 1)/\epsilon} - 1)}{(e^{\hat{\beta}/\varepsilon} - e^{(\hat{\beta} - t_f\varepsilon)/\varepsilon})}. \tag{4.40}$$

$$C_3 = \hat{\beta} - \varepsilon \ln \left(\frac{\varepsilon(1 - e^{(\hat{\beta} - 1)/\varepsilon})}{(e^{-t_f} - 1)} \right). \tag{4.41}$$

Since the argument of the logarithm in (4.41) is negative, C_3 is a complex number which can be written as

$$C_3 = \hat{\beta} - \varepsilon \ln \left(\frac{-\varepsilon(1 - e^{(\hat{\beta} - 1)/\varepsilon})}{(e^{-t_f} - 1)} \right) - i\pi\varepsilon \tag{4.42}$$

TABLE 3A

Example 3, $\varepsilon = 10^{-3}$

$t_f \rightarrow$	1	10	20	30		Absolute Rel. Error at $t_f=30$
x	y(x)	y(x)	y(x)	y(x)	$y(x)_{exact}$	
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	0.0
$2.5(10^{-4})$	$1.0500164(10^{-3})$	$1.5086461(10^{-3})$	$1.5086915(10^{-3})$	$1.5086915(10^{-3})$	$1.5086915(10^{-3})$	0.0
$1.0(10^{-3})$	0.0	$4.5862974(10^{-4})$	$4.5867514(10^{-4})$	$4.5867515(10^{-4})$	$4.5867515(10^{-4})$	0.0
$1.0(10^{-2})$	0.0	0.0	$4.5398899(10^{-8})$	$4.5400960(10^{-8})$	$4.5400960(10^{-4})$	0.0
$2.0(10^{-2})$	0.0	0.0	0.0	$2.0610600(10^{-12})$	$2.0611536(10^{-12})$	0.0
$3.0(10^{-2})$	0.0	0.0	0.0	0.0	$9.3564045(10^{-17})$	0.0
$4.0(10^{-2})$	0.0	0.0	0.0	0.0	0.0	0.0
$5.0(10^{-2})$	0.0	0.0	0.0	0.0	0.0	0.0
$1.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$2.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$3.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$4.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$5.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$6.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$7.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$8.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$9.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.0	0.0	0.0	0.0	0.0

TABLE 3B

Example 3, $\varepsilon = 10^{-4}$

$t_f \rightarrow$	1	10	20	30		Absolute Rel. Error at $t_f=30$
x	y(x)	y(x)	y(x)	y(x)	$y(x)_{exact}$	
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	0.0
$2.5(10^{-5})$	$1.0500164(10^{-4})$	$1.5086461(10^{-4})$	$1.5086915(10^{-4})$	$1.5086915(10^{-4})$	$1.5086915(10^{-4})$	0.0
$1.0(10^{-4})$	0.0	$4.5862974(10^{-5})$	$4.5867514(10^{-5})$	$4.5867515(10^{-5})$	$4.5867515(10^{-5})$	0.0
$5.0(10^{-4})$	0.0	$6.7153485(10^{-7})$	$6.7607474(10^{-7})$	$6.7607494(10^{-7})$	$6.7607494(10^{-7})$	0.0
$1.0(10^{-3})$	0.0	0.0	$4.5398899(10^{-9})$	$4.5400960(10^{-9})$	$4.5400960(10^{-9})$	0.0
$2.0(10^{-3})$	0.0	0.0	0.0	$2.0610600(10^{-13})$	$2.0611536(10^{-13})$	0.0
$3.0(10^{-3})$	0.0	0.0	0.0	0.0	$9.3564045(10^{-18})$	0.0
$4.0(10^{-3})$	0.0	0.0	0.0	0.0	0.0	0.0
$5.0(10^{-3})$	0.0	0.0	0.0	0.0	0.0	0.0
$5.0(10^{-2})$	0.0	0.0	0.0	0.0	0.0	0.0
$1.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$2.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$3.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$4.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$5.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$6.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$7.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$8.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$9.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.0	0.0	0.0	0.0	0.0

TABLE 3C

Example 3, $\varepsilon = 10^{-6}$

$t_f \rightarrow$	1	10	20	30		Absolute Rel. Error at t_f=30
x	y(x)	y(x)	y(x)	y(x)	$y(x)_{exact}$	
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	0.0
$2.5(10^{-7})$	$1.0500164(10^{-6})$	$1.5086461(10^{-6})$	$1.5086915(10^{-6})$	$1.5086915(10^{-6})$	$1.5086915(10^{-6})$	0.0
$5.0(10^{-7})$	$4.7407698(10^{-7})$	$9.3270673(10^{-7})$	$9.3275213(10^{-7})$	$9.3275213(10^{-7})$	$9.3275213(10^{-7})$	0.0
$1.0(10^{-6})$	$2.2204460(10^{-22})$	$4.5862974(10^{-7})$	$4.5867514(10^{-7})$	$4.5867515(10^{-7})$	$4.5867515(10^{-7})$	0.0
$2.0(10^{-6})$	0.0	$1.4536806(10^{-7})$	$1.4541346(10^{-7})$	$1.4541346(10^{-7})$	$1.4541346(10^{-7})$	0.0
$3.0(10^{-6})$	0.0	$5.1023780(10^{-8})$	$5.1069179(10^{-8})$	$5.1069181(10^{-8})$	$5.1069181(10^{-8})$	0.0
$5.0(10^{-6})$	0.0	$6.7153485(10^{-9})$	$6.7607474(10^{-9})$	$6.7607494(10^{-9})$	$6.7607494(10^{-9})$	0.0
$1.0(10^{-5})$	0.0	0.0	$4.5398899(10^{-11})$	$4.5400960(10^{-11})$	$4.5400960(10^{-11})$	0.0
$2.0(10^{-5})$	0.0	0.0	0.0	$2.0610602(10^{-15})$	$2.0611536(10^{-15})$	0.0
$3.0(10^{-5})$	0.0	0.0	0.0	0.0	$9.3564045(10^{-20})$	0.0
$4.0(10^{-5})$	0.0	0.0	0.0	0.0	0.0	0.0
$5.0(10^{-2})$	0.0	0.0	0.0	0.0	0.0	0.0
$1.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$2.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$3.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$4.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$5.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$6.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$7.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$8.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
$9.0(10^{-1})$	0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.0	0.0	0.0	0.0	0.0

and the real part of C_3, C_3^R is

$$C_3^R = \hat{\beta} - \varepsilon \ln \left(\frac{-\varepsilon(1 - e^{(\hat{\beta}-1)/\varepsilon})}{(e^{-t_f} - 1)} \right). \tag{4.43}$$

In Tables 3A, 3B, and 3C for $\alpha = 1$, $\beta = 0$, and $t_f = 1$, 10, 20, and 30 are listed the solutions by the boundary value method for $\varepsilon = 10^{-3}$, 10^{-4}, and 10^{-6}, respectively. As expected, for each ε, the solutions converge to the exact solution as t_f increases. Indeed for each ε at $t_f = 30$, the maximum relative error is zero.

Conclusions

We have described the boundary value technique for solving singular perturbation problems. It is a practical method, easily implemented on a computer to solve singular perturbation problems with a modest amount of problem preparation. We have illustrated the method with three examples with known solutions and have demonstrated that the boundary value method approximates the exact solution well.

Acknowledgments

The author acknowledges the helpful discussions with Dr. John Greenstadt on Example 3.

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